MODELING AND SIMULATION OF IMPACT AND CONTROL IN SOCIAL NETWORKS

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THE PLAN

MODELS OF INFLUENCE ON NETWORKS

French (1956), Harary (1959), De Groot (1974) –

a basic model of influence in a social group

Models of influence on networks: development

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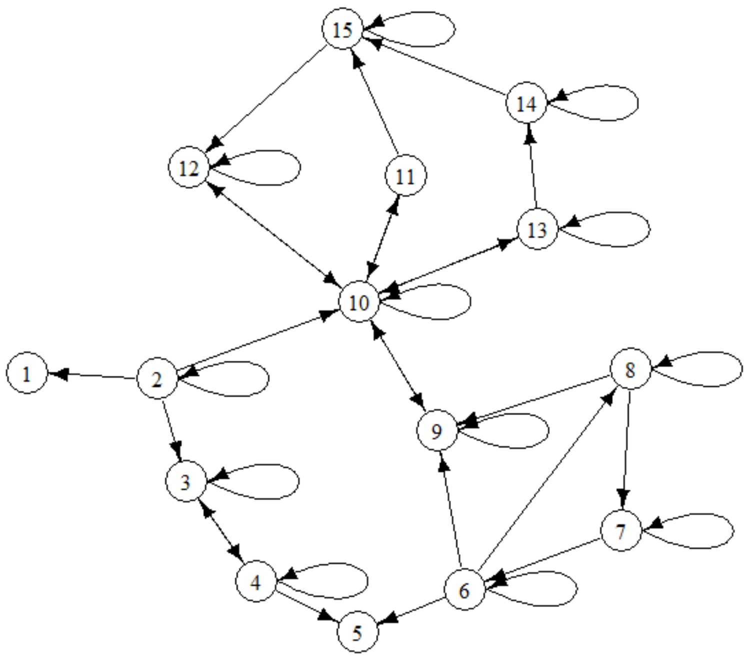
A BASIC MODEL OF INFLUENCE ON A NETWORK

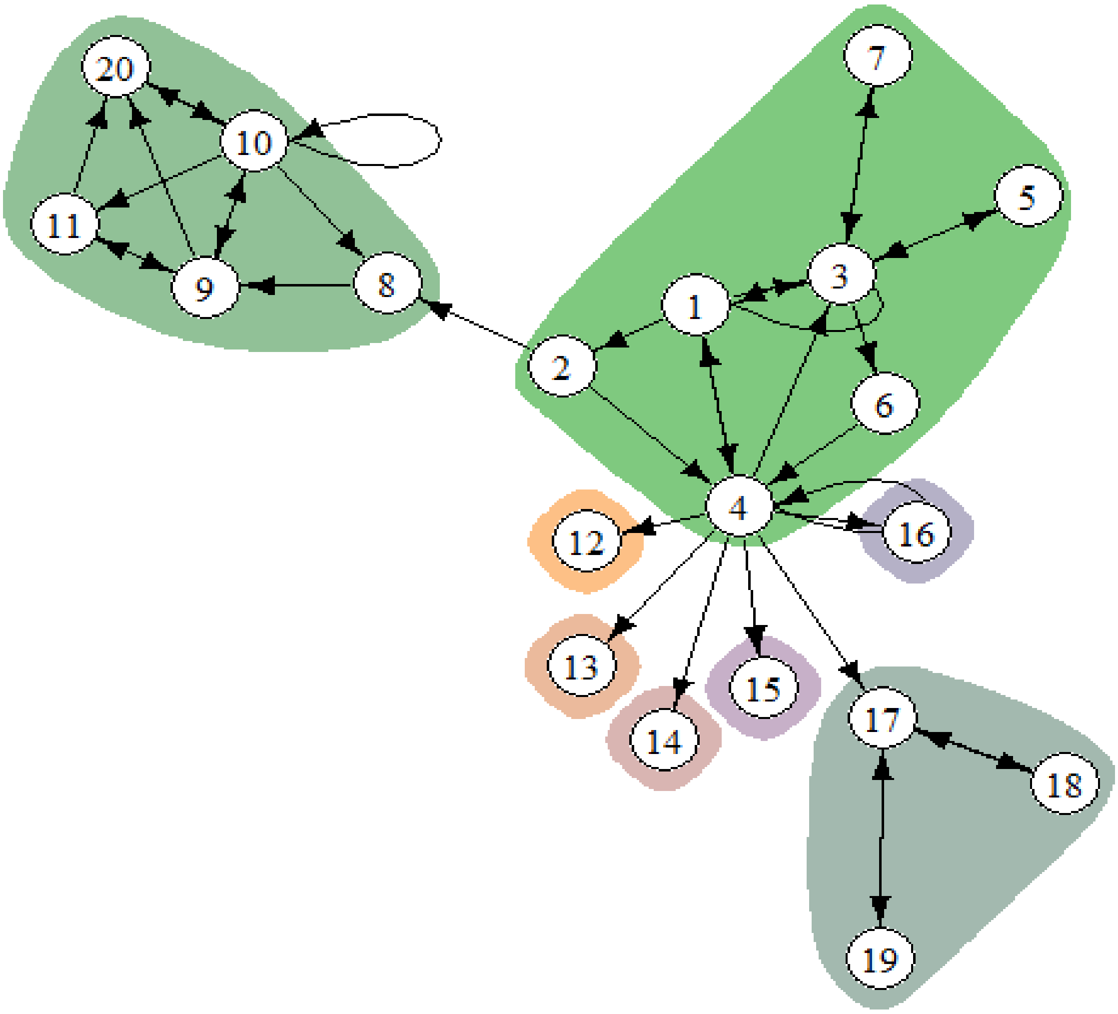


$$D = (Y, A),$$

$$Y = \{y_1, \dots, y_n\},$$

$$A = \| a_{ij} \|_{i,j=1}^{n}$$

STRONG SUBGROUPS AND SATELLITES



An influence digraph contains
two types of strong components:
those which **belong to the vertex base of the condensation** of this digraph and those **which do not.**
The strong components of the first type are called **the strong subgroups,** and other vertices are **satellites.**

The respective algorithm is implemented by standard procedures of the R programming language.

Opinions

The initial opinions for all vertices are given:

$$\mathbf{x^0 = (x_1{}^0,\ldots,x_n{}^0)}$$

The [natural] dynamics of opinions

is determined by the rule

$$x_j^{t+1} = \sum_{i=1}^{n} a_{ij} x_i^t,\ x_j^0 = x_{j0}, \qquad \textbf{(1)}$$

$$j = 1,\ldots,n.$$

A common stable final opinion of a strong subgroup:

$$x_i^{\infty} = \sum_{k=1}^{n_i} w_k^{(i)} x_{k0}^{(i)} \qquad \textbf{(2)}$$

A stable final opinion of the j-th satellite:

$$x_j^{\infty} = \sum_{i=1}^{r} b_{ji} x_i^{\infty} \qquad \textbf{(3)}$$

The respective original algorithm is implemented by the R programming language.

THE PRINCIPAL IDEA

So, all final opinions are determined only by the initial opinions of the members of the strong subgroups.

Thus, it is assumed to be rational for all control agents to exert their influence only to the members of strong subgroups (opinion leaders) which are determined in the stage of analysis of the network.

OPTIMAL CONTROL: PROBLEM FORMULATION

$$J = \sum_{t=1}^{T} e^{-\rho t}\left[\sum_{j=1}^{n} x_j^t - \sum_{k=1}^{m} u_k^t\right] \to \max \tag{4}$$

$$\sum_{t=1}^{T}\sum_{j=1}^{m} e^{-\rho t} u_j^t \le R\,; \tag{5}$$

$$x_j^{t+1} = b_j\sqrt{u_j^t} + \sum_{i=1}^{n} a_{ij}x_i^t,\ x_j^0 = x_{j0},\ j = 1,\ldots,n,\ t = 0,1,\ldots,T. \tag{6}$$

Here x_j^t - an opinion of the j-th agent in the instant t;

u_j^t - a control impact to the j-th agent;

a_{ij} - coefficient of interaction between i-th and j-th agents;

b_j - coefficient of the efficiency of impact to the j-th agent;

n - a total number of the target audience;

m - a number of the members of strong subgroups;

R - a marketing budget.

This is a discrete optimal control model with nonlinear dynamics. It was investigated by means of computer simulation.

COMPUTER SIMULATION EXPERIMENTS

Three model examples (15, 20, 100 vertices).

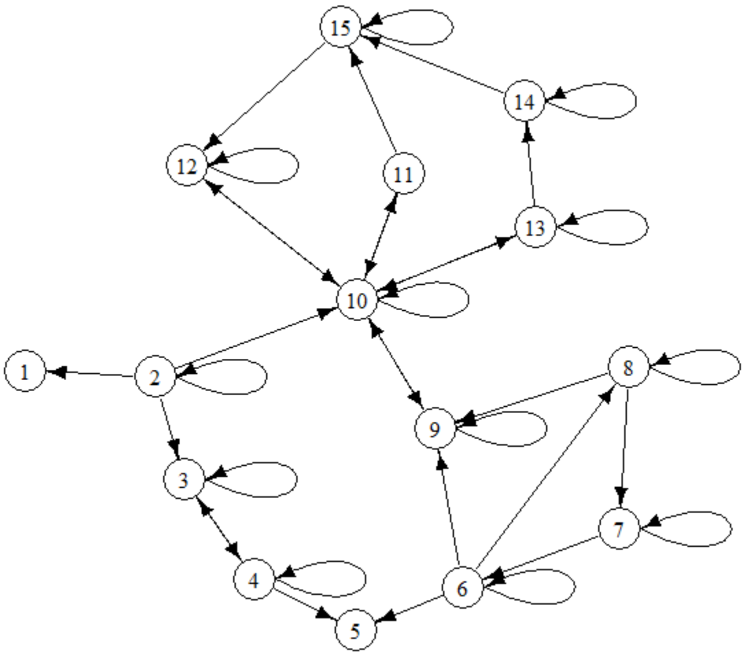
Scenarios of impact:

1. To all agents

2. To the members of strong subgroups only

Is the second variant sufficient?

MODEL EXAMPLE



R = 60 000 (budget)
T= 200, 500, 20000 (number of steps)
n =15 (total number of agents)
x0=(500, 900, 500, 400, 400, 500, 800, 500, 600, 400, 100, 200,200, 300, 500).

u_j = R/(Tn)

	T=200	T=500	T=20000

To all agents	x1=...=x4=1794;
	x5=1595;
	x6=x7=x8=1494;
	x9=1569;
	x10=...=x15=1719
	
	J=*283 253.3*
		x1=...=x4=2314;
		x5=2114;
		x6=x7=x8=2014;
		x9=2089;
		x10=...=x15=2239
		
		J= *272 897,1*
			x1=...=x4=29184;
			x5=28984;
			x6=x7=x8=28884;
			x9=28959;
			x10=...=x15=29109
			
			J= *251 515,4*

To strong subgroups only	x1=x2=2623;
	x3=2562; x4=2545;
	x5=2394;
	x6=x7=x8=2632;
	x9=2374; x10=2494;
	x11=2485; x12=2468;
	x13=2477; x14=2442;
	x15=2451
	
	J= *307 121,9*
		x1=3633; x2=3639;
		x3=3595; x4=3584;
		x5=3415;
		x6=x7=x8=3339;
		x9=3393; x10=3524;
		x11=3518; x12=3507;
		x13=3513; x14=3491;
		x15=3497
		
		J= *288 059,9*
			x1=55612; x2=55672;
			x3=x4=55670;
			x5=55471;
			x6=x7=x8=55372;
			x9=55466;
			x10=x11=55595;
			x12=55594;
			x13=55595;
			x14=x15=55594
			
			J= *252 273*

QUALITATIVELY REPRESENTATIVE SCENARIOS

(Ougolnitsky, Usov, 2018)

$QRS = \{u^1, u^2, \ldots u^m\}$ is a QRS set in the optimal control problem with precision Δ if:

(a) for any two elements of this set $u^i, u^j \in QRS$

$$| J^{(i)} - J^{(j)} | > \Delta ; \tag{7}$$

(b) for any element $u^l \notin QRS$ there is $u^j \in QRS$ such that

$$| J^{(l)} - J^{(j)} | \leq \Delta . \tag{8}$$

Check that T = {200,500, 200000} is a QRS set.

Step T	Payoff J
200	***2 937 211***
500	***2 917 487***
1000	2 906 170
20000	2 883 652
40000	2 881 675
60000	2 880 795
80000	2 880 269
100000	2 879 910
120000	2 879 645
140000	2 879 438
160000	2 879 272
180000	2 879 134
200000	***2 879 017***

Let Δ = 19000. Condition (7):

$$| J^{(200)} - J^{(500)} | = 19724 > \Delta$$

$$| J^{(200)} - J^{(200000)} | = 58194 > \Delta$$

$$| J^{(500)} - J^{(200000)} | = 38470 > \Delta$$

Condition (8):

$$| J^{(1000)} - J^{(500)} | = 11317 < \Delta$$

$$| J^{(20000)} - J^{(200000)} | = 4635 < \Delta$$

$$| J^{(40000)} - J^{(200000)} | = 2658 < \Delta$$

$$| J^{(60000)} - J^{(200000)} | = 1778 < \Delta$$

$$| J^{(80000)} - J^{(200000)} | = 1252 < \Delta$$

$$| J^{(100000)} - J^{(200000)} | = 893 < \Delta$$

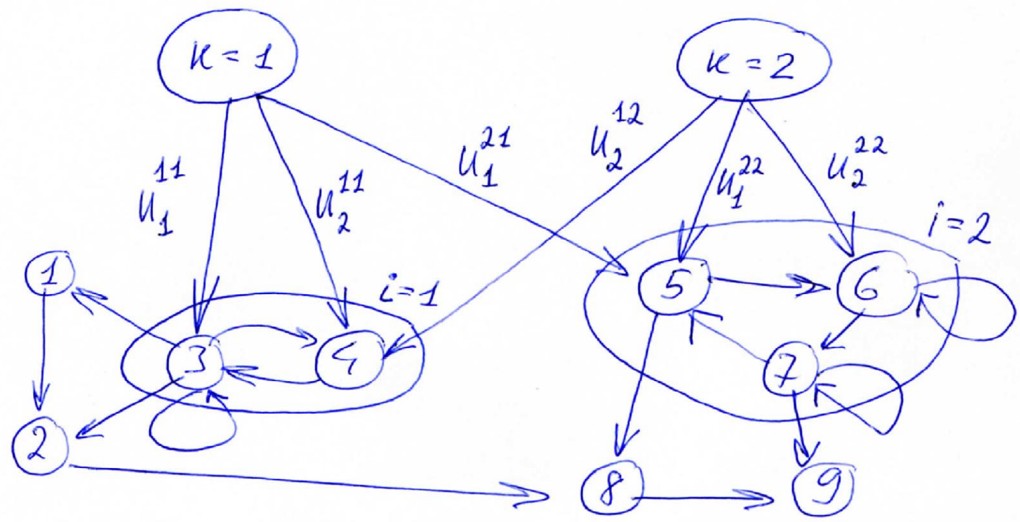
$$| J^{(120000)} - J^{(200000)} | = 628 < \Delta$$

$$| J^{(140000)} - J^{(200000)} | = 421 < \Delta$$

$$| J^{(160000)} - J^{(200000)} | = 255 < \Delta$$

$$| J^{(180000)} - J^{(200000)} | = 117 < \Delta$$

A MODEL EXAMPLE FOR GAMES



$n_1 = 2$ $\qquad$ $n_2 = 3$

$m_{11} = 2$ $\qquad$ $m_{12} = 1$

$m_{21} = 1$ $\qquad$ $m_{22} = 2$

$S_1 = \{3,4\} \rightarrow j \in \{1,2\}$ при $i = 1$

$S_2 = \{5,6,7\} \rightarrow j \in \{1,2,3\}$ при $i = 2$

STATIC GAME WITH INDEPENDENT PLAYERS

The objective is to increase the final opinions of the members of the target audience. The firms exert influence to the initial **opinions of some members of the strong subgroups**.

Each k-th firm solves the following problem:

$$J_k = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \left\{ w_j^{(i)} x_i^{\infty} \right\} \xrightarrow[u_{j0}^{(ik)} : h(i,j,k)=1]{} \max ,$$

with constraints

$$x_i^{\infty} = \sum_{j=1}^{n_i} w_j^{(i)} \left(x_{j0}^{(i)} + \sum_{l=1}^{m} u_{j0}^{(il)} h(i,j,l) \right),$$

$$\sum_{i=1}^{r} \sum_{j=1}^{n_i} \left[u_{j0}^{(ik)} h(i,j,k) \right]^p = R_k , \ k = 1,2,\ldots,m,$$

$$u_{j0}^{(ik)} \geq 0, \ 1 \leq i \leq r, \ 1 \leq j \leq n_i, \ 1 \leq k \leq m .$$

Here x_i - an opinion of the i-th agent;
$w_k^{(i)}$ – a component of the stationary vector of the Markov chain with the transitive matrix A_i^T
$u_{j0}^{(ik)}$ – a marketing impact to the initial opinion of the j-th member of the i-th strong subgroup by the k-th firm;
n_i – a total number of the i-th strong subgroup;
$h(i,j,k) = 1$, if k-th firm exerts influence to the j-th agent of the i-th strong subgroup;
$h(i,j,k) = 0$, otherwise;
R_k - the marketing budget of the k-th firm.

STATIC GAME WITH INDEPENDENT PLAYERS

The optimal controls have the form

$$u_{j^*0}^{(i^*k)} = \frac{\sqrt[p-1]{w_{j^*}^{(i^*)}}\,\sqrt[p]{R_k}}{\sqrt[p]{\sum\limits_{\substack{i,j:\\ h(i,j,k)=1}} \left[w_j^{(i)}\right]^{\frac{p}{p-1}}}}.$$

The final opinions are equal to:

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i} w_j^{(i)} x_{j0}^{(i)} + \sum_{k=1}^{m}\left\{\left(\sum_{\substack{i,j:\\ h(i,j,k)=1}} \left[w_j^{(i)}\right]^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \sqrt[p]{R_k}\right\}.$$

The total payoff is equal to

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i} w_j^{(i)} x_{j0}^{(i)} + \sum_{k=1}^{m}\left\{\left(\sum_{\substack{i,j:\\ h(i,j,k)=1}} \left[w_j^{(i)}\right]^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \sqrt[p]{R_k}\right\} - \sum_{k=1}^{m} R_k .$$

STATIC GAME WITH COOPERATIVE PLAYERS

$$J = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \left\{ w_j^{(i)} x_i^{\infty} \right\} - R \xrightarrow[u_{j0}^{(ik)}:\, h(i,j,k)=1]{} \max ,$$

(now k is not fixed as above)

with constraints

$$x_i^{\infty} = \sum_{j=1}^{n_i} w_j^{(i)} \left(x_{j0}^{(i)} + \sum_{l=1}^{m} u_{j0}^{(il)} h(i,j,l) \right),$$

$$\sum_{i=1}^{r} \sum_{j=1}^{n_i} \left[u_{j0}^{(ik)} h(i,j,k) \right]^p = R ,$$

$$u_{j0}^{(ik)} \geq 0 , \; 1 \leq i \leq r , \; 1 \leq j \leq n_i , \; 1 \leq k \leq m ,$$

where $R = \sum_{k=1}^{m} R_k$.

STATIC GAME WITH COOPERATIVE PLAYERS

The optimal controls are

$$u_{j^*0}^{\left(i^* k^*\right)} = \frac{\sqrt[p-1]{w_{j^*}^{\left(i^*\right)}}\,\sqrt[p]{R}}{\sqrt[p]{\sum_{k=1}^{m}\left(\sum_{\substack{i,j:\\ h(i,j,k)=1}}\left[w_j^{(i)}\right]^{\frac{p}{p-1}}\right)}}.$$

The final opinions are equal to:

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i} w_j^{(i)} x_{j0}^{(i)} + \left[\sum_{k=1}^{m}\left(\sum_{\substack{i,j:\\ h(i,j,k)=1}}\left[w_j^{(i)}\right]^{\frac{p}{p-1}}\right)\right]^{\frac{p-1}{p}} \sqrt[p]{R}\,.$$

The total payoff is equal to

$$\sum_{i=1}^{r}\sum_{j=1}^{n_i} w_j^{(i)} x_{j0}^{(i)} + \left[\sum_{k=1}^{m}\left(\sum_{\substack{i,j:\\ h(i,j,k)=1}}\left[w_j^{(i)}\right]^{\frac{p}{p-1}}\right)\right]^{\frac{p-1}{p}} \sqrt[p]{R} - R\,.$$

COMPARISON OF THE RESULTS

The ratio of optimal cooperative and independent solutions is equal to

$$\frac{u^C}{u^N} = \sqrt[p]{\frac{R \sum_{i,j:h(i,j,k)=1} [w_j^{(i)}]^{\frac{p}{p-1}}}{R_k \sum_{k=1}^{m} \sum_{i,j:h(i,j,k)=1} [w_j^{(i)}]^{\frac{p}{p-1}}}}$$

Thus,

from one side, cooperation decreases the marketing efforts because there are more firms and they in fact advertize the same product;

from the other side, the marketing efforts can increase because a joint marketing budget is greater in the case of cooperation.

DIFFERENCE GAME WITH INDEPENDENT PLAYERS

The objective in this model is **to maximize the sum of opinions of the members of the target audience in the whole period from *t=1* till *t=n*.** The firms exert an influence in closed-loop strategies to the current opinions of the members of the strong subgroups.

There are N agents and m firms. Each i-th firm solves the following optimization problem:

$$\sum_{t=1}^{n}\left[e^{-\rho(t-1)}\sum_{j=1}^{N}\left(x_j^t+\sum_{i=1}^{m}u_j^{i(t)}\left(x_j^t\right)\right)\right]\to\max,$$

$$x_j^{t+1}=\sum_{l=1}^{N}a_{lj}\left(x_l^t+\sum_{i=1}^{m}u_l^{i(t)}\left(x_l^t\right)\right), x_j^0=x_{j0},\ j=1,2,\ldots,N, t=1,2,\ldots,n-1,$$

$$\sum_{t=1}^{n}e^{-\rho(t-1)}\sum_{j=1}^{N}\left[u_j^{i(t)}\left(x_j^t\right)\right]^p=R_i,$$

$$u_j^{i(t)}\left(x_j^t\right)\geq 0,\ j=1,2,\ldots,N, t=1,2,\ldots,n,$$

where $u_j^{i(t)}\left(x_j^t\right)$ – a control impact of the i-th firm to the j-th agent in the t-th time period.

DIFFERENCE GAME WITH INDEPENDENT PLAYERS

It is more convenient to write the problem statement and the solution in the matrix form. The *i*-th firm's problem is

$$\varepsilon \sum_{t=1}^{n} \delta^{t-1}\left(X^t + Bu^{i(t)}\right) \to \max ,$$

$$X^{t+1} = A^T\left[X^t + Bu^{i(t)}\right],\ t=1,2,\ldots,n-1,\ X^1 = X ,$$

$$u^{i(t)} \geq 0,\ t=1,2,\ldots,n ,$$

$$\sum_{t=1}^{n} \delta^{t-1} \sum_{j=1}^{N}\left(u_j^{i(t)}\left(x_j^t\right)\right)^p = R_i .$$

The matrix *B* consists of *m* blocks (number of firms).

Each Boolean block B_i describes the influence of the *i*-th firm and has the dimension $N \times m_i$,

N - number of agents,

m_i - a total number of the influenced members of the strong subgroups.

X – a column vector of the state variables,

u – a column vector of controls,

ε – a row vector of N units,

$\delta = e^{-\rho}$ (discounting).

DIFFERENCE GAME WITH INDEPENDENT PLAYERS

The problem is solved recurrently. The optimal solution has the form

$$u_j^i = \sqrt[p]{\frac{R_i}{\sum\limits_{k=1}^{m_i}\left(1+\delta A_k+\delta^2 A_k^2+\ldots+\delta^{n-1}A_k^{n-1}\right)^{\frac{p}{p-1}}}}\left(1+\delta A_j+\delta^2 A_j^2+\ldots\delta^{n-1}A_j^{n-1}\right)^{\frac{1}{p-1}}.$$

The payoff of each i-th firm

$$\varepsilon\left[I+\delta A^T+\delta^2\left(A^T\right)^2+\ldots+\delta^{n-1}\left(A^T\right)^{n-1}\right]X+$$

$$+\sum_{i=1}^{m}\left\{\sqrt[p]{R_i}\left[\sum_{j=1}^{m_i}\sum_{s=0}^{n-1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\ldots+\delta^{n-2}\sum_{j=1}^{m_i}\sum_{s=0}^{1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\delta^{n-1}m_i\right]^{\frac{p-1}{p}}\right\}-R_i\,.$$

The total payoff is equal to

$$\varepsilon\left[I+\delta A^T+\delta^2\left(A^T\right)^2+\ldots+\delta^{n-1}\left(A^T\right)^{n-1}\right]X+$$

$$+\sum_{i=1}^{m}\left\{\sqrt[p]{R_i}\left[\sum_{j=1}^{m_i}\sum_{s=0}^{n-1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\ldots+\delta^{n-2}\sum_{j=1}^{m_i}\sum_{s=0}^{1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\delta^{n-1}m_i\right]^{\frac{p-1}{p}}\right\}-\sum_{i=1}^{m}R_i.$$

DIFFERENCE GAME WITH COOPERATIVE PLAYERS

There are N agents and m firms. A control body representing all firms solves the following optimal control problem:

$$\sum_{t=1}^{n}\left[e^{-\rho(t-1)}\sum_{j=1}^{N}\left(x_j^t+\sum_{i=1}^{m}u_j^{i(t)}\left(x_j^t\right)\right)\right]\to\max,$$

$$x_j^{t+1}=\sum_{l=1}^{N}a_{lj}\left(x_l^t+\sum_{i=1}^{m}u_l^{i(t)}\left(x_l^t\right)\right), x_j^0=x_{j0},\ j=1,2,\ldots,N,\ t=1,2,\ldots,n-1,$$

$$\sum_{t=1}^{n}\left\{e^{-\rho(t-1)}\sum_{i=1}^{m}\sum_{j=1}^{N}\left[u_j^{i(t)}\left(x_j^t\right)\right]^p\right\}=R,\ R=\sum_{i=1}^{m}R_i,$$

$$u_j^{i(t)}\left(x_j^t\right)\geq 0,\ j=1,2,\ldots,N,\ i=1,2,\ldots,m,\ t=1,2,\ldots,n,$$

where $u_j^{i(t)}\left(x_j^t\right)$ – an influence of the i-th firm to the j-th agent in the t-th time period.

DIFFERENCE GAME WITH COOPERATIVE PLAYERS

In the matrix form the problem is

$$\varepsilon \sum_{t=1}^{n} \delta^{t-1}\left(X^{t} + Bu^{i(t)}\right) \to \max ,$$

$$X^{t+1} = A^{T}\left[X^{t} + Bu^{i(t)}\right],\ t = 1, 2, \ldots, n-1,\ X^{1} = X ,$$

$$u^{i(t)} \geq 0,\ i = 1, 2, \ldots, m,\ t = 1, 2, \ldots, n,$$

$$\sum_{t=1}^{n} \delta^{t-1} \sum_{i=1}^{m} \sum_{j=1}^{N}\left(u_{j}^{i(t)}\left(x_{j}^{t}\right)\right)^{p} = R .$$

where T stands for transposition, and $\sum_{i=1}^{m} \sum_{j=1}^{N}\left(u_{j}^{i(t)}\left(x_{j}^{t}\right)\right)^{p}$ – a sum of controls of all firms in the t-th period.

DIFFERENCE GAME WITH COOPERATIVE PLAYERS

The optimal solution for the n-step game is

$$u_j^i = \sqrt[p]{\frac{R}{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m_i}\left(1+\delta A_k+\delta^2 A_k^2+\ldots+\delta^{n-1}A_k^{n-1}\right)^{\frac{p}{p-1}}}}\left(1+\delta A_j+\delta^2 A_j^2+\ldots\delta^{n-1}A_j^{n-1}\right)^{\frac{1}{p-1}}.$$

The total payoff is

$$\varepsilon\left[I+\delta A^T+\delta^2\left(A^T\right)^2+\ldots+\delta^{n-1}\left(A^T\right)^{n-1}\right]X-\sum_{i=1}^{m}R_i+$$

$$+\sqrt[p]{\sum_{i=1}^{m}R_i}\left\{\sum_{i=1}^{m}\left[\sum_{j=1}^{m_i}\sum_{s=0}^{n-1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\ldots+\delta^{n-2}\sum_{j=1}^{m_i}\sum_{s=0}^{1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\delta^{n-1}m_i\right]\right\}^{\frac{p-1}{p}}.$$

The difference between total payoffs in the cooperative and independent cases is equal to

$$\sqrt[p]{\sum_{i=1}^{m}R_i}\left\{\sum_{i=1}^{m}\left[\sum_{j=1}^{m_i}\sum_{s=0}^{n-1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\ldots+\delta^{n-2}\sum_{j=1}^{m_i}\sum_{s=0}^{1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\delta^{n-1}m_i\right]\right\}^{\frac{p-1}{p}}-$$

$$-\sum_{i=1}^{m}\left\{\sqrt[p]{R_i}\left[\sum_{j=1}^{m_i}\sum_{s=0}^{n-1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\ldots+\delta^{n-2}\sum_{j=1}^{m_i}\sum_{s=0}^{1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}}+\delta^{n-1}m_i\right]^{\frac{p-1}{p}}\right\}. \quad (*)$$

COMPARISON OF THE RESULTS

Lemma. When p>1 the expression (*) is non-negative.

Proof. Denote:

$$q = \frac{p}{p-1},\ S_i = \sum_{j=1}^{m_i}\sum_{s=0}^{n-1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}} + \ldots + \delta^{n-2}\sum_{j=1}^{m_i}\sum_{s=0}^{1}\left(\delta^s A_j^s\right)^{\frac{p}{p-1}} + \delta^{n-1} m_i .$$

Then the expression (*) takes the form

$$\left(\sum_{i=1}^{m} R_i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m} S_i\right)^{\frac{1}{q}} - \sum_{i=1}^{m}\left(R_i\right)^{\frac{1}{p}}\left(S_i\right)^{\frac{1}{q}} .$$

According to Holder's inequality if $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$ then

$$\sum_{i=1}^{m} a_i b_i \le \left(\sum_{i=1}^{m} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m} b_i^q\right)^{\frac{1}{q}},$$

or, what is the same,

$$\sum_{i=1}^{m}\left(a_i\right)^{\frac{1}{p}}\left(b_i\right)^{\frac{1}{q}} \le \left(\sum_{i=1}^{m} a_i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m} b_i\right)^{\frac{1}{q}} .$$

Setting in Holder's inequality $a_i = R_i$, $b_i = S_i$ gives:

$$\left(\sum_{i=1}^{m} R_i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m} S_i\right)^{\frac{1}{q}} - \sum_{i=1}^{m}\left(R_i\right)^{\frac{1}{p}}\left(S_i\right)^{\frac{1}{q}} \ge 0 .$$

Thus, from the point of view of the total payoff a cooperation is more advantageous than an independent behavior if the control costs are big, and vice versa, if they are small.

DIFFERENTIAL GAME WITH INDEPENDENT PLAYERS

Each i-th firm solves the problem:

$$J_i = \int_0^T e^{-\rho t} \sum_{j=1}^{N} x_j(t) dt \to \max ,$$

$$\dot{x}_j = \sum_{i=1}^{r} b_j^i \sqrt{u_j^i\left(x_j(t)\right)} + \sum_{l=1}^{N} a_{lj} x_l(t), \quad x_j(0) = x_{j0}, \; j = 1,2,\ldots,N ,$$

$$\sum_{j=1}^{N} u_j^i\left(x_j(t)\right) = R_i^t , \quad \int_0^T e^{-\rho t} R_i^t dt = R_i ,$$

where j – an index of agent,

i – an index of firm,

N – a number of agents,

r – a number of firms.

The Hamilton-Jacobi-Bellman equation has the form:

$$\rho V_i - \frac{\partial V_i}{\partial t} = \max_{u_j^i, 1 \le j \le N} \left\{ \sum_{j=1}^{N} x_j(t) + \sum_{j=1}^{N} \frac{\partial V_i}{\partial x_j} \left[\sum_{i=1}^{r} b_j^i \sqrt{u_j^i\left(x_j\right)} + \sum_{l=1}^{N} a_{lj} x_l \right] \right\} \tag{1}$$

s.t. $\sum_{j=1}^{N} u_j^i\left(x_j(t)\right) = R_i^t$.

Maximization by u_j^i, $j = 1,2,\ldots,N$, gives

$$\frac{\partial V}{\partial x_j} b_j^i \frac{1}{2} \left(u_j^i\right)^{-\frac{1}{2}} = \mu ,$$

where μ – Lagrange multiplier, then

$$\frac{\frac{\partial V}{\partial x_{j_2}} b_{j_2}^i}{\frac{\partial V}{\partial x_{j_1}} b_{j_1}^i} = \left(\frac{u_{j_2}^i}{u_{j_1}^i} \right)^{\frac{1}{2}}, \qquad \frac{u_{j_1}^i}{\left(b_{j_1}^i \frac{\partial V}{\partial x_{j_1}} \right)^2} \sum_{j=1}^{N} \left(b_j^i \frac{\partial V}{\partial x_j} \right)^2 = R_i^t \text{, thus}$$

$$u_j^i = \frac{R_i^t\left(b_j^i \frac{\partial V}{\partial x_j}\right)^2}{\sum_{j=1}^{N}\left(b_j^i \frac{\partial V}{\partial x_j}\right)^2}. \qquad (2)$$

Assume that Bellman's functions are linear

$$V_i(x,t) = \sum_{j=1}^{N} \alpha_j(t) x_j + \beta(t),$$

and write (1) as

$$\rho \sum_{j=1}^{N} \alpha_j(t) x_j + \rho\beta(t) - \sum_{j=1}^{N} \alpha'_j(t) x_j - \beta'(t) =$$

$$= \sum_{j=1}^{N} x_j(t) + \sum_{j=1}^{N}\sum_{l=1}^{N} \alpha_j(t) a_{lj} x_l + \sum_{i=1}^{r}\sum_{j=1}^{N} \alpha_j(t) b_j^i \frac{\sqrt{R_i^t}\,\alpha_j(t) b_j^i}{\sqrt{\sum_{j=1}^{N}\left(\alpha_j(t) b_j^i\right)^2}},$$

or

$$\rho \sum_{j=1}^{N} \alpha_j(t) x_j + \rho\beta(t) - \sum_{j=1}^{N} \alpha'_j(t) x_j - \beta'(t) =$$

$$= \sum_{j=1}^{N} x_j(t) + \sum_{j=1}^{N}\sum_{l=1}^{N} \alpha_j(t) a_{lj} x_l + \sum_{i=1}^{r} \sqrt{R_i^t} \sqrt{\sum_{j=1}^{N}\left(\alpha_j(t) b_j^i\right)^2}.$$

Equation of the coefficients for the same degrees of x gives:

$$\rho\alpha_j(t) - \alpha'_j(t) = 1 + \sum_{l=1}^{N} \alpha_l(t) a_{jl},$$

or

$$\alpha'_j(t) - \rho\alpha_j(t) + \sum_{l=1}^{N} \alpha_l(t) a_{jl} = -1 \quad , \qquad (3)$$

and

$$\rho\beta(t) - \beta'(t) = \sum_{i=1}^{r} \sqrt{R_i^t} \sqrt{\sum_{j=1}^{N}\left(\alpha_j(t) b_j^i\right)^2},$$

or

$$\beta'(t) - \rho\beta(t) = -\sum_{i=1}^{r} \sqrt{R_i^t} \sqrt{\sum_{j=1}^{N}\left(\alpha_j(t) b_j^i\right)^2} \quad . \qquad (4)$$

Rewrite the system (3) in a matrix form

$$\alpha' = (\rho I - A)\alpha - \varepsilon, \tag{5}$$

where A – an influence matrix, α – a column vector of the coefficients α_j, $j = 1,2,\ldots,N$, I – a unit matrix, ε – N-column of units.

The solution of (5) gives:

$$\bar{\alpha} = (\rho I - A)^{-1}\varepsilon, \quad \alpha = e^{(\rho I - A)t}C + (\rho I - A)^{-1}\varepsilon.$$

The boundary values have the form:

$$\alpha(T) = 0,$$

thus

$$C = -e^{-(\rho I - A)t}(\rho I - A)^{-1}\varepsilon,$$

then

$$\alpha = -e^{(\rho I - A)(t - T)}(\rho I - A)^{-1}\varepsilon + (\rho I - A)^{-1}\varepsilon = \left(e^{(A - \rho I)(T - t)} - I\right)(A - \rho I)^{-1}\varepsilon.$$

When $t = 0$ we have

$$\alpha(0) = \left(e^{(A - \rho I)T} - I\right)(A - \rho I)^{-1}\varepsilon. \tag{6}$$

The equation (4) is solved by the method of variation of parameters:

$$\beta(t) = e^{\rho t}C(t).$$

Then

$$\beta' = C'e^{\rho t} + C\rho e^{\rho t} - C\rho e^{\rho t} = -\sum_{i=1}^{r}\sqrt{R_i^t \sum_{j=1}^{N}\left(b_j^i\alpha_j(t)\right)^2},$$

or

$$C'e^{\rho t} = -\sum_{i=1}^{r}\sqrt{R_i^t \sum_{j=1}^{N}\left(b_j^i\alpha_j(t)\right)^2},$$

that gives

$$C(t) = -\sum_{i=1}^{r}\int_0^t e^{-\rho\tau}\sqrt{R_i^\tau \sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau + C.$$

As

$$C(T)=0,$$

we have

$$C=\sum_{i=1}^{r}\int_{0}^{T}e^{-\rho\tau}\sqrt{R_i^{\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau,$$

therefore,

$$C(t)=\sum_{i=1}^{r}\int_{t}^{T}e^{-\rho\tau}\sqrt{R_i^{\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau.$$

So,

$$\beta(t)=\sum_{i=1}^{r}\int_{t}^{T}e^{-\rho(\tau-t)}\sqrt{R_i^{\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau.$$

When $t=0$ we have

$$\beta(0)=\sum_{i=1}^{r}\int_{0}^{T}e^{-\rho\tau}\sqrt{R_i^{\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau.$$

Thus,

$$\max_{u_j^i,1\le j\le N} J_i=V_i\left(x(0),0\right)=\sum_{j=1}^{N}\alpha_j(0)x_j(0)+\sum_{i=1}^{r}\int_{0}^{T}e^{-\rho\tau}\sqrt{R_i^{\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau, \quad (7)$$

where $\alpha_j(0)$, $j=1,2,\ldots,N$, components of the vector $\alpha(0)$ (see (6)).
Now it is sufficient to solve an isoperimetric problem:

$$\int_{0}^{T}e^{-\rho\tau}\sqrt{R_i^{\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2}\,d\tau\to\max$$

s.t.

$$\int_{0}^{T}e^{-\rho\tau}R_i^{\tau}d\tau=R_i.$$

The Lagrange function is:

$$L\left(t,R_i^t,\left(R_i^t\right)',\lambda\right)=e^{-\rho t}\sqrt{R_i^t\sum_{j=1}^{N}\left(b_j^i\alpha_j(t)\right)^2}+\lambda e^{-\rho t}R_i^t.$$

The Euler's equation takes the form

$$\frac{\partial L\left(t, R_i^t, \left(R_i^t\right)', \lambda\right)}{\partial R_i^t} = \frac{e^{-\rho t}\sqrt{\sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2}}{2\sqrt{R_i^t}} + \lambda e^{-\rho t} = 0 .$$

So,

$$2\sqrt{R_i^t}\,\lambda = -\sqrt{\sum_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2} ,$$

and

$$R_i^t = \frac{\sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2}{4\lambda^2} .$$

The Lagrange multiplier is found from the budget constraint:

$$\frac{\int\limits_0^T e^{-\rho t} \sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2 dt}{4\lambda^2} = R_i .$$

We have

$$\frac{1}{4\lambda^2} = \frac{R_i}{\int\limits_0^T e^{-\rho t} \sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2 dt} ,$$

then

$$R_i^t = \frac{R_i \sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2}{\int\limits_0^T e^{-\rho t} \sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2 dt} .$$

Finally,

$$\beta(0) = \sum_{i=1}^{r}\int_0^T e^{-\rho\tau}\sqrt{R_i}\,\frac{\sqrt{\sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(\tau)\right)^2}\sqrt{\sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(\tau)\right)^2}}{\sqrt{\int\limits_0^T e^{-\rho t} \sum\limits_{j=1}^{N}\left(b_j^i \alpha_j(t)\right)^2 dt}}\, d\tau =$$

$$= \sum_{i=1}^{r} \sqrt{R_i} \frac{\int_0^T e^{-\rho\tau} \sum_{j=1}^{N} \left(b_j^i \alpha_j(\tau)\right)^2 d\tau}{\sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt}} = \sum_{i=1}^{r} \sqrt{R_i} \sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt} .$$

The payoff of each player is equal to

$$\sum_{j=1}^{N} x_{j0} \alpha_j(0) + \sum_{i=1}^{r} \sqrt{R_i} \sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt} ,$$

where $\alpha_j(0)$ are components of the vector (6), $j = 1, 2, \ldots, N$.

The substitution of the value for R_i^t in (2) gives the final form of the optimal solution:

$$u_j^i = \frac{\left(b_j^i \alpha_j(t)\right)^2}{\sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2} \cdot \frac{R_i \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2}{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt} = \frac{R_i \left(b_j^i \alpha_j(t)\right)^2}{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt} ,$$

$i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, N$.

COOPERATIVE BEHAVIOR

The optimal control problem

$$J = \int_0^T e^{-\rho t} \sum_{j=1}^N x_j(t) dt \to \max,$$

$$\dot{x}_j = \sum_{i=1}^r b_j^i \sqrt{u_j^i\left(x_j(t)\right)} + \sum_{l=1}^N a_{lj} x_l(t), \quad x_j(0) = x_{j0}, \quad j = 1,2,\ldots,N,$$

$$\sum_{i=1}^r \sum_{j=1}^N u_j^i\left(x_j(t)\right) = R^t, \quad \int_0^T e^{-\rho t} R^t dt = R = \sum_{i=1}^r R_i,$$

is solved similarly, and the optimal control has the form

$$u_j^i = \frac{\left(b_j^i \frac{\partial V}{\partial x_j}\right)^2}{\sum_{i=1}^r \sum_{j=1}^N \left(b_j^i \frac{\partial V}{\partial x_j}\right)^2} \cdot \frac{R \sum_{i=1}^r \sum_{j=1}^N \left(b_j^i \alpha_j(t)\right)^2}{\int_0^T e^{-\rho t} \sum_{i=1}^r \sum_{j=1}^N \left(b_j^i \alpha_j(t)\right)^2 dt} = \frac{R\left(b_j^i \alpha_j(t)\right)^2}{\int_0^T e^{-\rho t} \sum_{i=1}^r \sum_{j=1}^N \left(b_j^i \alpha_j(t)\right)^2 dt},$$

$i = 1,2,\ldots,r$; $j = 1,2,\ldots,N$.

The total cooperative payoff is equal to

$$\sum_{j=1}^N x_{j0} \alpha_j(0) + \sqrt{R} \sqrt{\int_0^T e^{-\rho \tau} \sum_{i=1}^r \sum_{j=1}^N \left(b_j^i \alpha_j(\tau)\right)^2 d\tau} - R,$$

and the total payoff in the independent case is equal to

$$\sum_{j=1}^N x_{j0} \alpha_j(0) + \sum_{i=1}^r \sqrt{R_i} \sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^N \left(b_j^i \alpha_j(t)\right)^2 dt} - \sum_{i=1}^r R_i.$$

COMPARISON OF INDEPENDENT AND COOPERATIVE SOLUTIONS

Thus, we should compare the terms

$$\sum_{i=1}^{r} \sqrt{R_i} \sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt}$$

for the independent behavior and

$$\sqrt{R} \sqrt{\int_0^T e^{-\rho\tau} \sum_{i=1}^{r} \sum_{j=1}^{N} \left(b_j^i \alpha_j(\tau)\right)^2 d\tau}$$

for the cooperative behavior.

According to Holder's inequality, for r pairs of positive numbers u_i, v_i, $i = 1,2,\ldots,r$

$$\sum_{i=1}^{r} u_i v_i \le \left(\sum_{i=1}^{r} (u_i)^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{r} (v_i)^q \right)^{\frac{1}{q}},$$

if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and it holds

$$\sum_{i=1}^{r} u_i v_i \ge \left(\sum_{i=1}^{r} (u_i)^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{r} (v_i)^q \right)^{\frac{1}{q}},$$

if $p < 1$, $p \ne 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Denote

$$u_i = \sqrt{R_i}\,,\ v_i = \sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt}$$

Given $p = q = 2$, according to Holder's inequality

$$\sum_{i=1}^{r} \sqrt{R_i} \sqrt{\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt} \le \sqrt{R} \sqrt{\int_0^T e^{-\rho\tau} \sum_{i=1}^{r} \sum_{j=1}^{N} \left(b_j^i \alpha_j(\tau)\right)^2 d\tau}\,.$$

Thus, the total payoff in the case of independent behavior is less or equal than in the case of cooperation.

INEQUALITY CONSTRAINTS

Now consider the problem with inequality constraints

$$J_i = \int_0^T e^{-\rho t}\left(\sum_{j=1}^N x_j(t) - \sum_{j=1}^N s_j^i u_j^i\left(x_j(t)\right)\right)dt \to \max,$$

$$s_j^i = \begin{cases} 1, & \text{если} \quad b_j^i \neq 0, \\ 0, & \text{если} \quad b_j^i = 0, \end{cases}$$

$$\dot{x}_j = \sum_{i=1}^r b_j^i \sqrt{u_j^i\left(x_j(t)\right)} + \sum_{l=1}^N a_{lj} x_l(t),\ x_j(0) = x_{j0},\ j = 1,2,\ldots,N,$$

$$\sum_{j=1}^N u_j^i\left(x_j(t)\right) \leq R_i^t,\ \int_0^T e^{-\rho t} R_i^t dt \leq \Re_i,$$

where R_i^t, $\Re_i$ - the respective budget constraints.

We will solve first the problem with **equality constraints**

$$J_i = \int_0^T e^{-\rho t}\left(\sum_{j=1}^N x_j(t) - \sum_{j=1}^N s_j^i u_j^i\left(x_j(t)\right)\right)dt \to \max,$$

$$s_j^i = \begin{cases} 1, & \text{если} \quad b_j^i \neq 0, \\ 0, & \text{если} \quad b_j^i = 0, \end{cases}$$

$$\dot{x}_j = \sum_{i=1}^r b_j^i \sqrt{u_j^i\left(x_j(t)\right)} + \sum_{l=1}^N a_{lj} x_l(t),\ x_j(0) = x_{j0},\ j = 1,2,\ldots,N,$$

$$\sum_{j=1}^N u_j^i\left(x_j(t)\right) = R_i^t,\ \int_0^T e^{-\rho t} R_i^t dt = \Re_i$$

for an arbitrary R_i, $0 \leq R_i \leq \Re_i$, and then choose the optimal value of R_i.

ANALYSIS

We have $\sum_{i=1}^{r} \Re_i = \Re$.

If for the *i*-th firm in the case of independent behavior

$$\int_0^T e^{-\rho t} \sum_{j=1}^{N} \left(b_j^i \alpha_j(t)\right)^2 dt \le 4\Re_i \quad \text{then}$$

we will say that the *i-th firm has enough resources,* otherwise the *i-th firm lacks resources.*

Similarly, in the case of cooperative behavior if

$$\int_0^T e^{-\rho\tau} \sum_{i=1}^{r} \sum_{j=1}^{N} \left(b_j^i \alpha_j(\tau)\right)^2 dt \le 4\Re \quad \text{then}$$

we will say that *all firms have enough resources*, otherwise *all firms lack resources*.

RESOURCES

Denote:

for independent behavior

$\Re_i$ - a resource allocated to the *i*-th firm;

R_i - a resource used by this firm in the optimal solution;

for cooperative behavior

$\Re$ - a total resource allocated to all firms

R - a total resource used by them in the optimal point.

In the case of independent behavior,

if an i-th firm lacks resources then $\Re_i = R_i$, and the firm's payoff depends monotonously on $\Re_i$.

If the i-th firm has enough resources then $\Re_i > R_i$, its payoff does not depend on R_i, the resource is redundant, and its shadow price μ is equal to zero.

Similarly, in the case of cooperative behavior,

if all firms lack resources then $\Re = R$, and the total payoff depends monotonously on $\Re$.

If all firms have enough resources then $R<\Re$, the total payoff does not depend on $\Re$, the resources are redundant, and the shadow price μ is equal to zero.

CONCLUSIONS FOR INEQUALITY CONSTRAINTS

Four cases are possible from the point of view of resource allocation.

1) All firms have enough resources and they are allocated such that each firm has enough resources. In this case the total payoff of all firms at their independent behavior is equal to

$$\sum_{j=1}^{N} x_{j0}\alpha_j(0)+\frac{1}{4}\sum_{i=1}^{r}\int_{0}^{T} e^{-\rho\tau}\sum_{j=1}^{N}\left(b_j^i\alpha_j(\tau)\right)^2 d\tau,$$

and it coincides with the total cooperative payoff.

2) All firms have enough resources but they are allocated such that some firms lack resources. In this case, the total resource used by all firms with independent behavior, is less than the respective total resource used in cooperative behavior, and the total payoff of all firms for their independent behavior is also less than the total cooperative payoff because there is a monotone dependence of payoffs from resource in the interval of the lack of resource.

3) All firms lack resources but some firms have enough resources. In this case, the total resource used by all firms with independent behavior, is less than the respective total resource used in cooperative behavior, and the total payoff of all firms for their independent behavior is also less than the total cooperative payoff.

4) Each firm lacks resources: therefore, all firms lack resources too. In this case, the total payoff of all firms for their independent behavior is less than the total cooperative payoff according to the above lemma.

MARKETING INTERPRETATION

Model element	**Mathematical sense**	**Marketing interpretation**
Basic agent	Network node	Segment of audience
Control agent	Network node	Market participants (firms), advertising agencies, mass media
Opinion of basic agent	Real value associated with each node (basic agent) that varies in time	Agent’s monthly (annual) expenses on a firm’s products
Trust (influence)	Arc between initial and terminal nodes	Word-of-mouth, other communications of agents
Degree of trust of basic agent to another one	Real value associated with each arc in a network	Quantitative characteristic of trust
Resulting opinion	Limit value of opinion over infinite time horizon	Stable resulting opinion over long period of time
Strong subgroup	Non-degenerate strong component of the network correspondent to an ergodic set of the respective Markov chain	Determines its own resulting opinions and the dependent opinions of other agents (**opinion leaders**)
Satellite	Subset of nodes representing degenerate strong components	Resulting opinions are completely determined by strong subgroups
Influence on opinions	Additive term of opinion vector (more complex cases are possible)	Marketing plan
Impact on degrees of trust (influence)	Additive term of influence matrix (more complex cases are possible)	Marketing plan
Goal of control	Domain in the state space of a network	Range of desired opinions

MARKETING INTERPRETATION

Model problems	Applications to marketing
Network analysis	1. Target audience segmentation, identification of strong subgroups that determine the inner common resulting opinions of subgroups members and also the individual resulting opinions of other agents (satellites) as a linear combination of resulting opinions of strong subgroups. 2. Calculation of centrality, prestige and other characteristics of the target audience.
Prediction on networks	Calculation of resulting opinions of all agents without external impact.
Optimal control on networks	Choice of optimal marketing actions (impact) for the target audience by one firm
Dynamic games on networks (conflict control)	Choice of compromise impact on the target audience in the case of competition and/or cooperation of firms (in the latter case, taking into account the homeostasis conditions, e.g., limited consumption).

GENERAL CONCLUSIONS

1. The models of influence on networks can be extended to optimization and game theoretic models. In this case it is rational for all control agents to exert their influence only to the members of strong subgroups which are determined in the stage of analysis of the network (an essential economy).

2. A system of descriptive, optimization, and game theoretic models provides a comprehensive analysis of social networks and their practical applications. For example, a marketing interpretation of the models is given.

3. The algorithms of network analysis and calculation of the final opinions are implemented in R programming language and tested on model examples.

4. The models of optimal control in marketing networks are studied by computer simulation, and the main research hypothesis was confirmed.

5. The analytical solutions are found for the statements of a static game, a difference game, and a differential game with constraints in the form of equalities/inequalities. A special method of solution of the problem with inequality constraints is proposed.

6. A comparative analysis of the solutions for the cases of independent and cooperative behavior of the players is made, and the marketing interpretation is given.

7. In most cases the cooperation is more advantageous than the independent behavior. In the same time, there are some more detailed conclusions. In the static game, from one side, cooperation decreases the marketing efforts because there are more firms and they in fact advertize the same product; from the other side, the marketing efforts can increase because a joint marketing budget is greater in the case of cooperation. In the difference game, from the point of view of the total payoff a cooperation is more advantageous than an independent behavior if the control costs are big, and vice versa, if they are small.

8. Some conclusions about the optimal marketing resource allocation are received for a differential game model.