Voter model for electoral and census data

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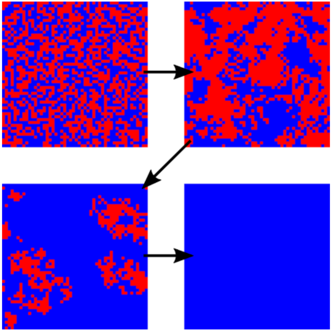
Theoretical premise

Classic voter model

Originally voter model was defined similarly to a cellular automaton:

- Agents are the cells of a two dimensional grid.
- Each agent is in one of the two states: $+1$ or -1.
- During each time step:
  - agent (A) is selected,
  - its neighbor (B) is selected,
  - A copies the state of B.



Temporal evolution of the classic voter model.

Original paper: Clifford & Sudbury, Biometrika 60: 581–588 (1973).

Picture collage from: http://rf.mokslasplius.lt/voter-model/

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Stable fixed points in opinions dynamics is not a thing

Therefore modifications to promote “instability”:

- Changing topology – it is possible to delay convergence.
- Non-binary states – it is possible to delay convergence.
- Panel discussions (q-Voter model) – should work in some cases.
- Independent transitions – works, but is N dependent.
- Agents with fixed state (zealots) – works, but is N dependent.
- **Non-extensive interactions (Kirman model) – works**.

In the last three cases model converges to a stationary Beta or Beta–binomial distribution.

q-Voter model: Castellano *et al.*, PRE 80: 041129 (2009)

Kirman model: Kirman, QJE 108: 137-156 (1993), Kononovicius & Ruseckas, EPJ B 87: 169 (2014).

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Ising model

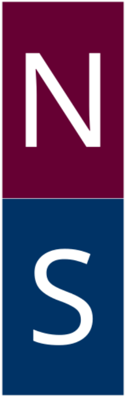
(why voter model is interesting to a socio–physicist)

Total energy of a magnetic system:

$$\mathcal{H} = -\frac{1}{2}\sum_{j\neq i} J_{i,j}\vec{\sigma}_i\vec{\sigma}_j - \vec{H}\sum_{i}\vec{\sigma}_i,$$

here $J_{i,j}$ is interaction constant, $\vec{\sigma}_i$ is spin of i-th particle, $\vec{H}$ is an external magnetic field.

- If $J_{i,j} \geq 0$, then the state with all parallel spins is the most probable.
- If $T = 0$, then the said state is a stable fixed point.



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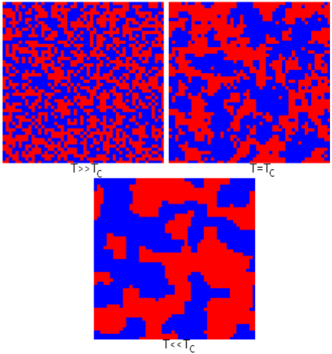
Glauber dynamics of the Ising model

- Particles are placed on a grid.
- Each particle has a magnetic spin of $+1$ or -1.
- During each time step:
  - a particle is selected,
  - its energy in both possible states ($+1$ or -1) is evaluated,
  - the final state is selected according to the Boltzmann (exponential) distribution.



Distinct phases of the Ising model with Glauber dynamics.

Image source: http://rf.mokslasplius.lt/ising-model/

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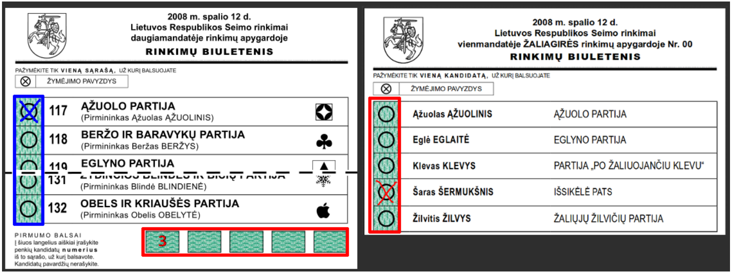
Empirical context

Empirical data



Example ballots. We are interested in political party performance (blue).

- LT parliamentary elections. Held each 4 years. Data is available at polling station level.
- UK census. Taken each decade. Data is available at various spatial levels.

Image source: Central Electoral Commission

Data sources: CEC (full, in Lithuanian), GitHub (partial, in English); NOMIS (UK census data)

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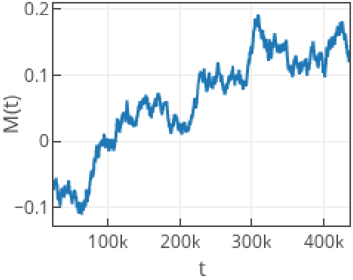
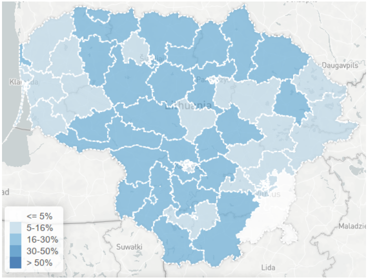
Models and Data





Models focus on temporal dynamics (left) while data is observed spatially (right).

Image sources: (left) http://rf.mokslasplius.lt/q-voter-model/, (right) http://rinkimurezultatai.lt

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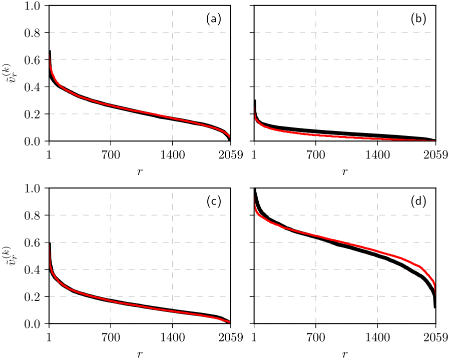
Simplest solution: Independent compartments

in Kirman's model (voter model with non-extensive interactions)



Rank–size distribution of the vote shares for 4 main parties in the Lithuanian Seimas 1992 elections: multi–state Kirman model (red) vs data (black).

Figure: Kononovicius, Complexity 2017: 7354642 (2017)

Similar works: (Sano *et al.*, 2017), (Braha & de Aguiar, 2017), (Fenner *et al.*, 2017)

Complicated solution: Commuting patterns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Is the Voter Model a Model for Voters?

Juan Fernández-Gracia, Krzysztof Suchecki, José J. Ramasco, Maxi San Miguel, and Víctor M. Eguíluz

Phys. Rev. Lett. **112**, 158701 – Published 18 April 2014; Erratum Phys. Rev. Lett. **113**, 089903 (2014)

Physics See Focus story: Voter Model Works for US Elections

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More

Article | References | Citing Articles (62) | Supplemental Material | PDF | HTML | Export Citation

ABSTRACT

The voter model has been studied extensively as a paradigmatic opinion dynamics model. However, its ability to model real opinion dynamics has not been addressed. We introduce a noisy voter model (accounting for social influence) with recurrent mobility of agents (as a proxy for social context), where the spatial and population diversity are taken as inputs to the model. We show that the dynamics can be described as a noisy diffusive process that contains the proper anisotropic coupling topology given by population and mobility heterogeneity. The model captures statistical features of U.S. presidential elections as the stationary vote-share fluctuations across counties and the long-range spatial correlations that decay logarithmically with the distance. Furthermore, it recovers the behavior of these properties when the geographical space is coarse grained at different scales—from the county level through congressional districts, and up to states. Finally, we analyze the role of the mobility range and the randomness in decision making, which are consistent with the empirical observations.

Issue

Vol. 112, Iss. 15 — 18 April 2014

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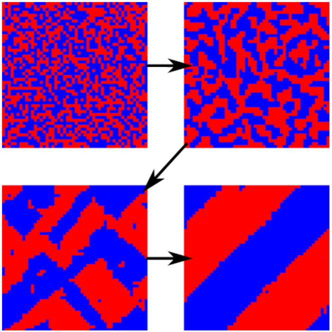
Fernandez–Garcia *et al.*, PRL 112: 158701 (2014)

Inspiration for a "third way" approach

Kawasaki dynamics of the Ising model

- Particles are placed on a grid.
- Each particle has a magnetic spin of $+1$ or -1. The spin is **conserved**.
- During each time step:
  - a particle is selected,
  - its neighbor is selected,
  - energy of the system is evaluated: (1) if particles remain where they are, and (2) if particles swap places.
  - the final state is selected according to the Boltzmann (exponential) distribution.



Temporal evolution of the Ising model with Kawasaki dynamics ($T \ll T_c$).

Collage from: a future post on Physics of Risk

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Compartmental voter model

Based on arXiv: 1906.01842 [physics.soc-ph] (accepted to J. Stat. Mech.)

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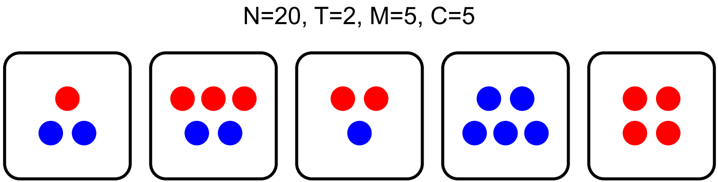
Static setup

- Consider N agents of T types.
- Let agent types be fixed.
- Let agents move between M compartments of capacity C.



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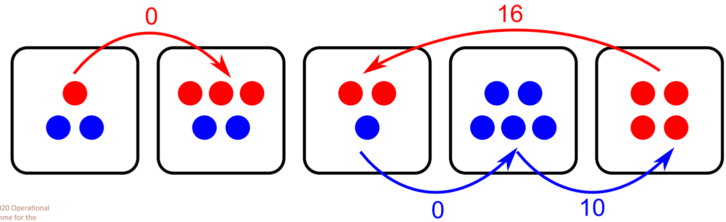
Dynamics

Let movement rates, from i to j for type k:

$$\lambda_{(k)}^{i\to j} = \begin{cases} X_i^{(k)}\left(\varepsilon^{(k)} + X_j^{(k)}\right) & \text{if } i \neq j \text{ and } N_j < C, \\ 0 & \text{otherwise,} \end{cases}$$

here $X_i^{(k)}$ is the number of type k agents in i, $\varepsilon^{(k)}$ is independent transition rate, while N_j is the number of all agents in j.

N=20, T=2, M=5, C=5, ε=2



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Infinite capacity ($C = N$)

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Stationary distribution

Given $C = N$ one can write closed form expression for the total entry and exit rates:

$$\lambda_{(k)}^{i+} = \sum_{j=1}^{M} \lambda_{(k)}^{j \rightarrow i} = \left[N^{(k)} - X_i^{(k)} \right] \left(\varepsilon^{(k)} + X_i^{(k)} \right),$$

$$\lambda_{(k)}^{i-} = \sum_{j=1}^{M} \lambda_{(k)}^{i \rightarrow j} = X_i^{(k)} \left([M-1]\, \varepsilon^{(k)} + \left[N^{(k)} - X_i^{(k)} \right] \right).$$

Which means that $X_i^{(k)}$ is distributed according to Beta-binomial distribution in this particular case.

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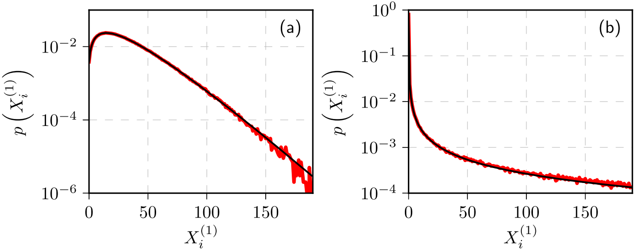
Numerical verification of stationary distribution



Model (red curves): $N = 3000$, $T = 1$, $M = 100$ and $C = N$ (all cases), $\varepsilon^{(1)} = 2$ (a) and 0.03 (b). Beta-binomial fit (black curves): $N = 3000$, $\alpha = \varepsilon^{(1)}$ and $\beta = (M - 1)\,\varepsilon^{(1)}$ (all cases).

Here $X_i^{(1)}$ is observed over time (i is fixed).

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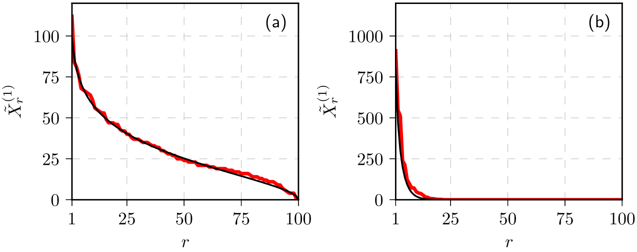
Numerical inquiry into rank–size distribution



Same simulation as in the previous slide: $N = 3000$, $T = 1$, $M = 100$ and $C = N$ (all cases), $\varepsilon^{(1)} = 2$ (a) and 0.03 (b).

Beta–binomial distribution provides a rather good fit as if compartments would be truly independent.

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Here $X_r^{(1)}$ is observed over compartments (r is variable).

Finite capacity ($C < N$)

Stationary distribution

- In general we don't know closed forms of the total entry and exit rates.
- Compartmental model is actually a multivariate finite–state Markov chain.
- In theory we can reduce any finite–state Markov chain to one–dimensional Markov chain by relabeling states.
- Alternatively, we can use the detailed balance condition.

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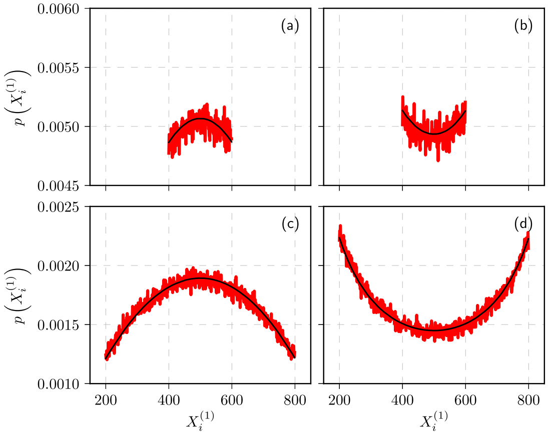
$T = 1$ and $M = 2$ (temporal)



Model (red): $N = 1000$ (all), $C = 600$ ((a) and (b)) and 800 ((c) and (d)), $\varepsilon^{(1)} = 2$ ((a) and (c)) and 0.03 ((b) ir (d)). Truncated Beta–binomial distribution (black): $N = 1000$, $\alpha = \varepsilon$, $\beta = (M - 1)\,\varepsilon$ (all).

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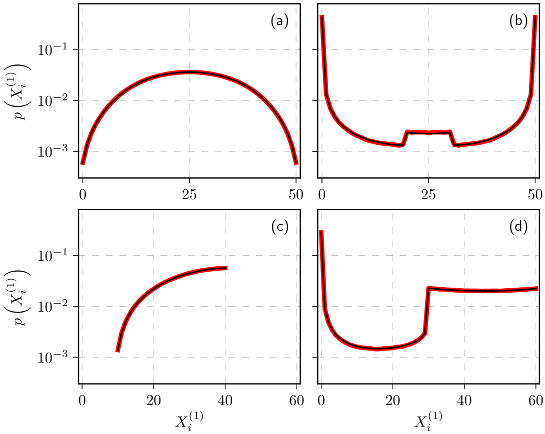
$M = 2$ and $T = 2$, $M = 3$ and $T = 1$ (temporal)



Model (red): $N = 100$, $M = 2$ and $T = 2$ ((a) and (b)), $N = 90$, $M = 3$ and $T = 1$ ((c) and (d)), $C = 40$ (c), 60 ((a) and (d)) and 80 (b), $\varepsilon = 2$ ((a) and (c)) and 0.03 ((b) and (d)). Analytical result obtained via Markov chains (black).

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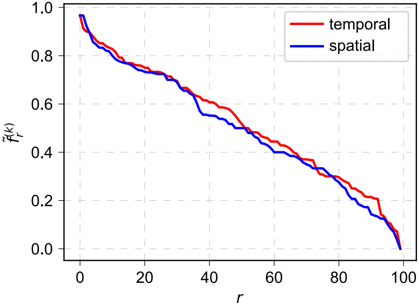
Spatio–temporal symmetry in rank–size distributions



Parameters: $N = 2600$, $T = 2$, $M = 100$, $C = 30$ and $\varepsilon = 2$

Note that here we consider $f_i^{(k)} = X_i^{(k)}/N_i$.

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Empirical examples

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Model parameter list

- T – number of types (from data).
- $N^{(k)}$ – total number of type k agents in the system (from data).
- M – number of compartments (from data).
- C – capacity of compartments.
- $\varepsilon^{(k)}$ – independent transition rate for type k agents.

In total the model has $2T + 3$ parameters:

- $T + 2$ parameters are obtained directly from data,
- $T + 1$ must be fitted.

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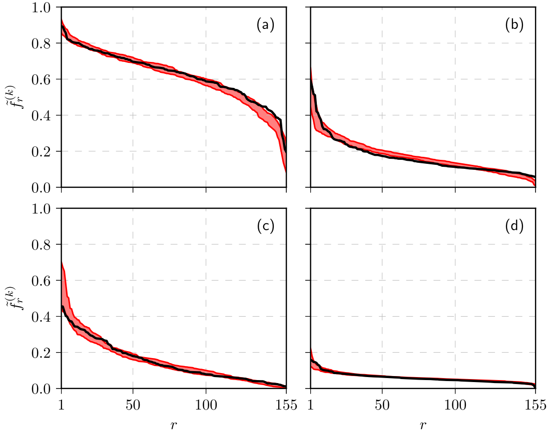
Ethnic groups in London (UK census 2011)



Considered groups (black curves): (a) White, (b) Asian, (c) Black, (d) other. Model (red areas): $N^{(w)} = 48515$, $N^{(a)} = 12865$, $N^{(b)} = 11470$ and $N^{(o)} = 4495$ ($N = 77345$), $\varepsilon^{(w)} = 2.5$, $\varepsilon^{(a)} = 4$, $\varepsilon^{(b)} = 1.5$, $\varepsilon^{(o)} = 15$, $M = 155$, $C = 600$.

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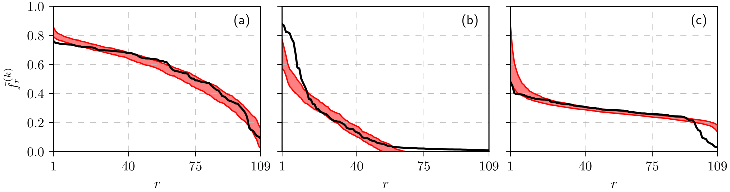
Red areas show 95% confidence intervals.

Religious groups in Leicester (UK census 2011)



Considered groups (black curves): (a) Christians, (b) no religion, (c) other. Model (red areas): $N^{(c)} = 30411$, $N^{(n)} = 8829$ and $N^{(o)} = 15151$ ($N = 54391$), $\varepsilon^{(c)} = 2.5$, $\varepsilon^{(n)} = 0.01$, $\varepsilon^{(o)} = 50$, $M = 109$, $C = 600$.

Red areas show 95% confidence intervals.

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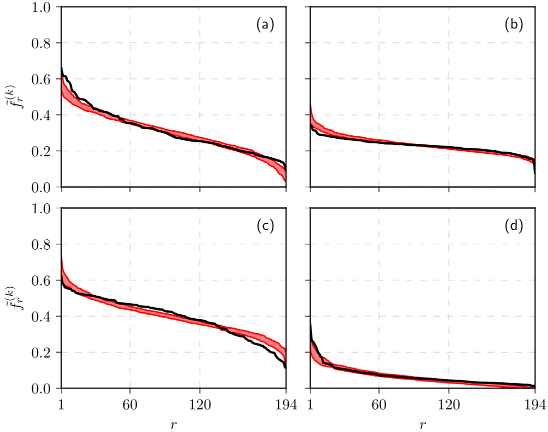
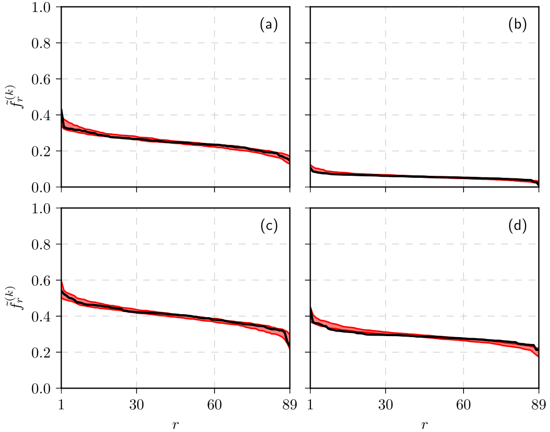
Working class in Sheffield (UK census 2011)



Considered groups (black curves): (a) higher, (b) intermediate and (c) lower occupations, (d) unemployed. Model (red areas): $N^{(1)} = 29876$, $N^{(2)} = 22310$, $N^{(3)} = 38218$ and $N^{(u)} = 6596$ ($N = 97000$), $\varepsilon^{(1)} = 3$, $\varepsilon^{(2)} = 50$, $\varepsilon^{(3)} = 12$, $\varepsilon^{(u)} = 2$, $M = 194$, $C = 600$.



Considered groups (black curves): (a) Sajudzio koalicija, (b) Lietuvos krikscioniu demokartu partija, (c) Lietuvos demokratine darbo partija, (d) other. Model (red areas): $N^{(s)} = 11125$, $N^{(l)} = 2581$, $N^{(d)} = 17978$ and $N^{(o)} = 12816$ ($N = 44500$), $\varepsilon^{(s)} = \varepsilon^{(l)} = \varepsilon^{(d)} = 25$, $\varepsilon^{(o)} = 75$, $M = 89$, $C = 600$.

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To conclude...

Some key points and outlying questions

- We have proposed the compartmental voter model, which is not a model for voters (in contrast to Fernandez-Garcia *et al.*).
- The proposed model reproduces census and electoral data rather well.
- Demographic processes are likely reason for spatial electoral heterogeneity.

- Does spatio–temporal symmetry hold for any finite birth–death process or is this property unique to the voter model?
- Does voter model on a grid produce similar results?
- How does this model compare to human mobility models?

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This talk was based on a preprint

Compartmental voter model

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Abstract

Numerous models in opinion dynamics focus on the temporal dynamics within a single spatial unit (e.g., country). While the opinions are often observed across multiple spatial units (e.g., polling stations) at a single point in time (e.g., elections). Aggregates of these observations, while quite useful in many applications, neglect the underlying spatial heterogeneity in opinions. To address this issue we build a simple compartmental agent–based model in which all agents have fixed opinions, but are able to change their compartments. We demonstrate that this model is able to generate compartmental rank–size distributions consistent with the empirical data.

1 Introduction

Most well–known models of opinion dynamics seem to imply that a steady state, either consensus or polarization, is inevitable [1–7]. However, local and spatial heterogeneity and ongoing exchange of opinions and cultural traits is a characterizing feature of social systems. Variety modifications of the well–known models were proposed to account for these features, such as inflexibility [8,9] or spontaneous flipping [10,11]. Some of the models were modified to account for the theories from the social sciences [12,13]. Effects of these modifications are still being actively reconsidered in context of network theory, non–linearity, complex contagion and applications towards financial markets [14–24]. Nevertheless even these modified models assume that opinion dynamics occur and

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Available online as arXiv: 1906.01842 [physics.soc-ph] (accepted to J. Stat. Mech.).

Thank you!

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Appendix: Obtaining stationary distribution

through detailed balance condition

Let $T = 1$ and $M = 2$. Detailed balance condition:

$$p\left(X_i^{(k)}\right)\lambda_{i+}^{(k)}\left(X_i^{(k)}\right) = p\left(X_i^{(k)}+1\right)\lambda_{i-}^{(k)}\left(X_i^{(k)}+1\right).$$

Lets rearrange:

$$p\left(X_i^{(k)}+1\right) = \frac{\lambda_{i+}^{(k)}\left(X_i^{(k)}\right)}{\lambda_{i-}^{(k)}\left(X_i^{(k)}+1\right)} p\left(X_i^{(k)}\right).$$

This can be solved recurrently from $X_i^{(k)} = N - C$ to C. It can be easily verified that the solution is truncated Beta–binomial distribution.

Appendix: Obtaining stationary distribution

by using Markov chains

- Let $N = 10, T = 2, M = 2, C = 6$.
- Then the state vector is $\left\{X_1^{(1)}, X_1^{(2)}\right\}$.
- List all possible state vectors: $\{0, 4\}, \ldots, \{5, 1\}$.
- Relabel: $\{0, 4\} \to 0, \{0, 5\} \to 1, \ldots$
- Write down transition matrix and find its eigensystem.
- Obtain the desired result:
  $P(X_1^{(1)} = 0) = P(\{0, 4\}) + P(\{0, 5\}) + P(\{0, 6\})$.

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Appendix: What about rank–size distributions?

While M is small, there is not much point to consider rank–size distributions. Though we can look from a different perspective.

- Let $N = 11$, $T = 1$, $M = 2$ and $C = 9$.
- State vector is $\left\{ X_1^{(1)} \right\}$.
- Possible states: $\{2\}, \ldots, \{9\}$.
- Consider evolution of state vector as a Markov chain.
- Use the fact that for rank–size distributions some of the states are symmetric (e.g., $\{2\}$ and $\{9\}$).

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Appendix: Spatio–temporal symmetry in RSDs

Parameters: $N = 11$, $T = 1$, $M = 2$, $C = 9$ and $\varepsilon = 3$.

X	Analytical	Spatial	Temporal
2 or 9	0.1730	0.1787	0.1723
3 or 8	0.2358	0.2397	0.2391
4 or 7	0.2830	0.2815	0.2835
5 or 6	0.3082	0.3001	0.3051

The catch is that we are interested in $T > 1$ and $M \gg 1$. Also We are interested not in X, but in $f_i^{(k)} = X_i^{(k)}/N_i$.

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