Birth-death processes in the modeling of opinion dynamics of social systems

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Outline

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Two alternatives in the modeling of social systems

Power-law statistics is a characteristic feature of complex social systems. Two alternatives to deal with this prevail.

Agent-based models (ABM), Microscopic bottom up

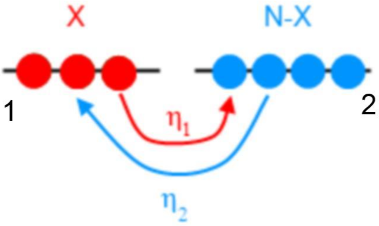
- Heterogeneous agents,
- Bounded rationality,
- Opinion dynamics,
- Birth-death processes.

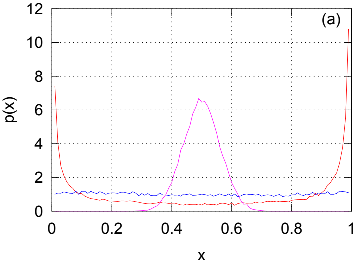
Stochastic models (Macroscopic-Phenomenological)

- ARCH, GARCH, etc.
- Standard model, stochastic volatility
- Macroscopic outcome of ABM.

The search of consentaneous (unanimous) approach in the modeling of social systems is our primary interest. More specifically the correspondence between birth-death processes and their macroscopic outcome expressed by stochastic differential equations (SDE).

An example of nonextensive birth-death process





Global interaction case

$$\eta_1(X,N) = \sigma_1 + h(N-X)$$
$$\eta_2(X,N) = \sigma_2 + hX$$

$$\mathrm{d}x = [\sigma_1(1-x) - \sigma_2 x]\,\mathrm{d}t + \sqrt{2hx(1-x)}\mathrm{d}W,$$

where $x = \frac{X}{N}$ and $N \to \infty$.
$P_0(x) \sim x^{\frac{\sigma_1}{h}-1}(1-x)^{\frac{\sigma_2}{h}-1}$

In the local interaction case the stochastic term vanishes. It is possible to derive Bass diffusion and Voter models.

Standard model - phenomenological (stochastic) consideration of finance

Standard model of price - Geometric Brownian motion

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma(t) S \mathrm{d}W.$$

$\sigma(t)$ - Endogenous volatility, the macroscopic outcome of agent based system fluctuations, agent based Markov jump process in the three state herding model.

W - Exogenous noise: Here considered as the market information flow or order flow noise. The basic assumption is that W fluctuations are much faster than $\sigma(t)$.

μ - the trend of exogenous noise, we ignore this trend as consider statistics of return not price.

$$r_\delta(t) = \sigma_t \omega_t, \quad \text{(ARCH, GARCH, ...)}$$

$$\sigma(t) = b(1 + a|p(t)|), \quad p(t) = \ln \frac{P(t)}{P_f(t)} = \frac{1-n_f(t)}{n_f(t)} \xi(t).$$

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The class of non-linear SDE with power law statistics and spurious long-range memory

$$\mathrm{d}x = (\eta - \tfrac{\lambda}{2})x^{2\eta-1}\mathrm{d}t + x^{\eta}\mathrm{d}W$$

$$P(x) \sim x^{-\lambda}, \quad S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 1 + \frac{\lambda - 3}{2\eta - 2}$$

$$\mathrm{d}x = \left(\eta - \tfrac{\lambda}{2}\right)(1 + x^2)^{\eta-1}x\mathrm{d}t + (1 + x^2)^{\frac{\eta}{2}}\mathrm{d}W$$

Publications

- Kaulakys, B.; Gontis, V. & Alaburda, M. (2005), *Phys. Rev.E*, 71, 051105.
- Kaulakys, B.; Ruseckas, J.; Gontis, V. & Alaburda, M. (2006), *Physica A*, 365, p. 217-221.
- Ruseckas, J. & Kaulakys B. (2014), *Journal of Statistical Mechanics*, P06004.

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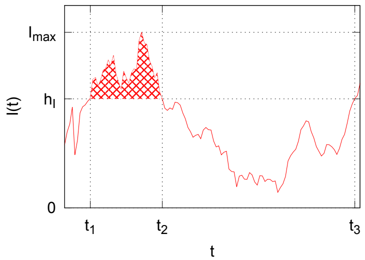
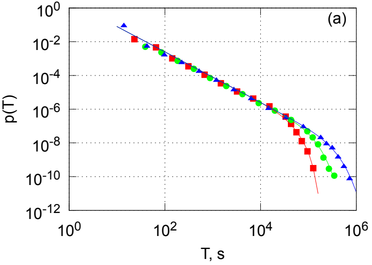
Burst and inter-burst duration and first passage time as a method to detect spurious long-range memory

We investigate stochastic time series as one dimensional processes. The Hurst exponent H is a key parameter, which can be easily detected from the power-law in PDF $P(T) \sim T^{2-H}$. Note, this PDF and H are invariant regarding nonlinear transformations of the time series and threshold.





Outline

Macroscopic SDE for the general birth-death process

$$\text{Birth rate - } \lambda(m,N) = (N-m)(\varepsilon_1 + m)$$
$$\text{Death rate - } \mu(m,N) = m(\varepsilon_2 + (N-m))$$

$$x = m/N, \quad \Delta x = 1/N, \quad N \to \infty$$

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\Delta x(\lambda-\mu)\right]P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\Delta x^2(\lambda+\mu)\right]P(x,t)$$

$$\mathrm{d}x = \lim_{N\to\infty}\Delta x(\lambda-\mu)\mathrm{d}t + \sqrt{\lim_{N\to\infty}\Delta x^2(\lambda+\mu)}\mathrm{d}W$$

$$\mathrm{d}x = \left[\varepsilon_1(1-x) - \varepsilon_2 x\right]\mathrm{d}t + \sqrt{2x(1-x)}\mathrm{d}W$$

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Definition of Bessel-like birth-death process

Let us to partition the interval $0 \leq z \leq \pi/\sqrt{2}$ into equally spaced intervals $\Delta z = \pi/\sqrt{2}/N$.

Lets define Bessel-like discrete birth-death process

$$\lambda_b(m, N) = \frac{N^2}{\pi^2}(1 + \frac{\varepsilon - 1/2}{m})$$

$$\mu_b(m, N) = \frac{N^2}{\pi^2}(1 - \frac{\varepsilon - 1/2}{m})$$

The passage time τ_m PDF as solution of continuous Bessel process

$$P^{(\nu)}(\tau_m) = \frac{z_m^{\nu-2}}{z_{m-1}^{\nu}} \sum_{k=1}^{\infty} \frac{j_{\nu,k} J_\nu\left(\frac{z_{m-1}}{z_m} j_{\nu,k}\right)}{J_{\nu+1}(j_{\nu,k})} \exp\left(-\frac{j_{\nu,k}^2}{2z_m^2}\tau_m\right),$$

where $P^{(\nu)}(\tau_m)$ is a probability density function of the first passage times at level z_m of Bessel process with index $\nu = \varepsilon - 1$ starting from z_{m-1}, J_ν is a Bessel function of the first kind of order ν and $j_{\nu,k}$ is a k-th zero of J_ν.

The Bounded Bessel-like birth-death process

We do seek to extend the applications of continuous PDF for the wider class of birth-death processes and first of all for the cases with bounded diffusion.

$$\lambda_{bb}(m,N)=\frac{N^2}{\pi^2}\left(1+\frac{(\varepsilon_1-1/2)}{m}-\frac{(\varepsilon_2-1/2)}{N-m}\right)$$

$$\mu_{bb}(m,N)=\frac{N^2}{\pi^2}\left(1-\frac{(\varepsilon_1-1/2)}{m}+\frac{(\varepsilon_2-1/2)}{N-m}\right),$$

It is useful to write the continuous SDE corresponding to this new non-extensive birth-death process for simplicity in the case $\varepsilon=\varepsilon_1=\varepsilon_2$

$$\mathrm{d}z=\frac{(\pi/\sqrt{2}-2z)(\varepsilon-1/2)}{z(\pi/\sqrt{2}-z)}\mathrm{d}t+\mathrm{d}W.$$

Basic transformations of SDE

Invariant regarding transformation $y \to \frac{1}{y}$ equation giving $1 < \beta < 2$ for the ratio $y = \frac{x}{1-x}$ in herding model

$$\mathrm{d}y = (2-\varepsilon_2)y(y+1)\mathrm{d}t + \sqrt{2y}(y+1)\mathrm{d}W$$

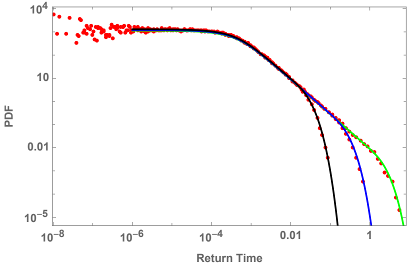
Lamperti transformation $z(y) = \sqrt{2}\arctan\sqrt{y}$ or $z(x) = \sqrt{2}\arcsin\sqrt{x}$

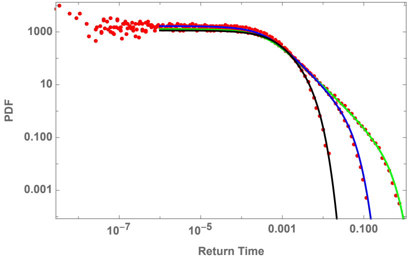
$$\mathrm{d}z = \frac{2\varepsilon - 1}{\sqrt{2}}\cot(z\sqrt{2})\mathrm{d}t + \mathrm{d}W$$

$0 \le x \le 1; \quad 0 \le y \le \infty; \quad 0 \le z \le \pi/\sqrt{2}; \quad 0 \le m \le N$ In the limit

$z \to 0 \qquad \mathrm{d}z = \frac{2\varepsilon-1}{2z}\mathrm{d}t + \mathrm{d}W$

Numerical comparison of PDF in discrete case



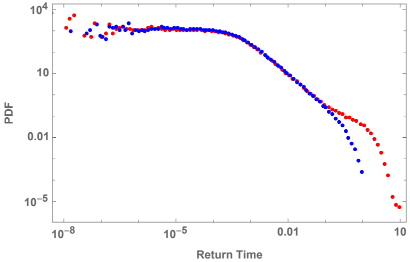


Numerical PDF versus continuous with $k_m = \frac{2m}{\pi}$ exponential terms.

$\varepsilon = 1.5$,
$m = 90$, Green
$m = 30$, Blue
$m = 10$, Black

$\varepsilon = 5.5$,
$m = 90$, Green
$m = 30$, Blue
$m = 10$, Black

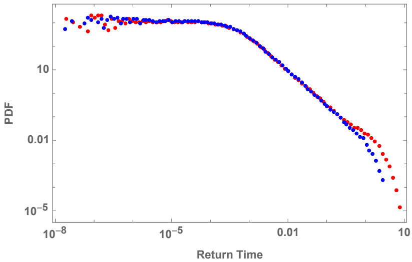
Numerical comparison of Bessel B-D process with bounded one



Numerical PDFs of Bessel process (Blue) with symmetrically bounded (Red).

$k_m = \frac{2m}{\pi}$

$\varepsilon = 3.5, \quad m = 70.$



$\varepsilon = 1.5, \quad m = 70,$

A simple method to describe the tail of PDF for the bounded B-D process

Lets account the tail of PDF as one more exponential term, then we get

$$P_{bb}(\tau_m) = (1-\rho)P_b(\tau_m) + \frac{\rho}{\tau_{m0}} \exp\left(-\frac{\tau_m}{\tau_{m0}}\right),$$

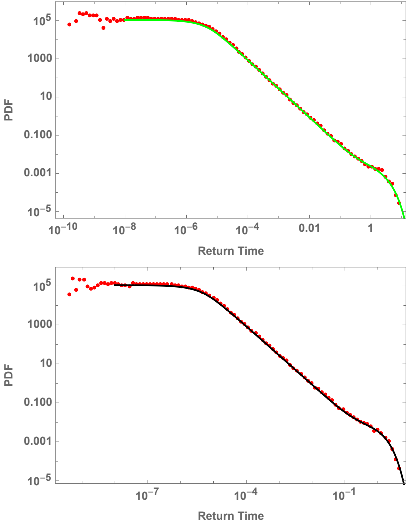
Then two equations for needed parameters are as follows

$$\tau_{m,1} = (1-\rho)Q_1 + \rho\tau_{m0}$$

$$\tau_{m,2} = (1-\rho)Q_2 + 2\rho\tau_{m0}^2$$

Here $\tau_{m,1}$ and $\tau_{m,2}$ are the first and second moments of τ_m, well defined from the rates of B-D process. Q_1 and Q_2 are coefficients defined as limited sums of Bessel functions.

Comparison of Bounded Bessel PDF with numerical one



Numerical PDFs of symmetrically Bounded Bessel process (Red points) with theoretical PDFs (Green and Black lines).

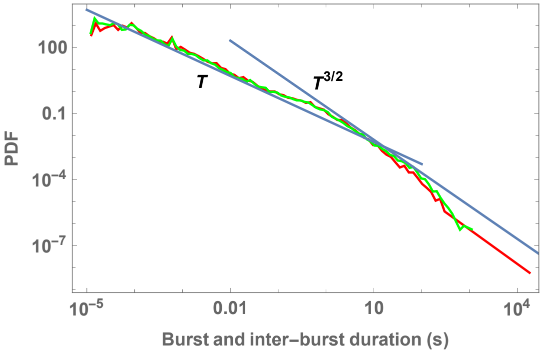
$N = 1000 \quad k_m = \frac{2m}{\pi}$,

$\varepsilon = 1.5, \quad m = 800.$

$\varepsilon = 1.5, \quad m = 70,$

Outline

Burst and inter-burst duration histogram for the time series of demand in financial markets calculated from the order book data



Demand is defined as $\ln \frac{bid(buy)}{ask(sell)}$. Green line - burst duration PDF. Red line - inter-burst duration PDF. Straight lines - guide the eye according power-laws.

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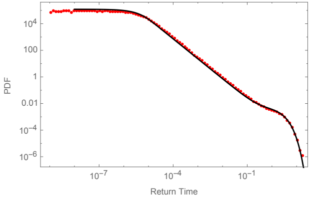
Conclusion 1: PDF of burst and inter-burst duration in the birth-death processes has fundamental power-law component 3/2 and universal cutoffs for small and large values of duration.



This is in consistence with first passage time theory for continuous description of Bessel process by SDE and well defined moments of birth-death processes.

$$\mathrm{d}z = \frac{2\varepsilon - 1}{2z}\mathrm{d}t + \mathrm{d}W,$$

A. Kononovicius, V. Gontis, *Approximation of the first passage time distribution for the birth–death processes*, J. Stat. Mech. (2019) 073402.

Conclusion 2: Empirical analyzes and agent based modeling suggest that long-range memory in finance and other social systems might origin from the birth-death processes - Markov chains, thus with high probability is spurious.

Thank You!